

C_0 -HILBERT MODULES

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ABSTRACT. We provide the definition and fundamental properties of algebraic elements with respect to an operator satisfying hypothesis (h) . Furthermore, we analyze Hilbert modules using C_0 -operators relative to a bounded finitely connected region Ω in the complex plane.

INTRODUCTION

The theory of contractions of class C_0 was developed by Sz.-Nagy-Foias [7], Moore-Nordgren [6], and Bercovici-Voiculescu [2,3], and J.A. Ball introduced the class of C_0 -operators relative to a bounded finitely connected region Ω in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves. The theory of Hilbert modules over function algebras has been developed by Ronald G. Douglas and Vern I. Paulsen [4].

We analyze Hilbert modules using C_0 -operators relative to Ω . Every operator T defined on a Hilbert space H satisfying hypothesis (h) is not a C_0 -operator relative to Ω . Thus, we provide the definition of an *algebraic element with respect to T* .

If B is the set of algebraic elements with respect to T , and it is closed, then naturally we have a bounded operator T_B from the quotient space H/B to H/B . In section 2, we discuss the relationships between the algebraic elements with respect to T_B in H/B and the algebraic elements with respect to T in H .

In section 3, we define a module action on a Hilbert space H by using a C_0 -operator T relative to Ω , and introduce a C_0 -Hilbert module H_T . Naturally, this raises the following question :

If every element of H_T is algebraic with respect to T over A , then T is either a C_0 -operator or not.

In this paper, we consider a case in which the rank of the C_0 -Hilbert module H_T is finite, and we show that if a generating set

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$\{h_1, \dots, h_k\} (k < \infty)$ of a Hilbert module H_T over A is contained in B , then T is a C_0 -operator.

Furthermore, if B is closed, then by using the Jordan model of a C_0 -operator T relative to Ω , we show that there are locally maximal C_0 -submodules $M_i (i = 0, 1, 2, \dots)$ of H_T such that $M_0 \subset M_1 \subset M_2 \subset \dots$.

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1. PRELIMINARIES AND NOTATION

1.1. Hilbert Modules. Let X be a compact, separable, metric space and let $C(X)$ denote the algebra of all continuous complex-valued functions on X . A *function algebra* on X is a closed subalgebra of $C(X)$, which contains the constant functions and separates points of X .

Definition 1.1. Let F be a function algebra, and let H be a Hilbert space. We say that H is a *Hilbert module over F* if there is a separately continuous mapping $\phi : F \times H \rightarrow H$ in each variable satisfying :

- (a) $\phi(1, h) = h$,
 - (b) $\phi(fg, h) = \phi(f, \phi(g, h))$,
 - (c) $\phi(f + g, h) = \phi(f, h) + \phi(g, h)$,
 - (d) $\phi(f, \alpha h + \beta k) = \alpha \phi(f, h) + \beta \phi(f, k)$,
- for every f, g in F , h, k in H , and α, β in \mathbb{C} .

We will denote $\phi(f, h)$ by $f.h$. For f in F , we let $T_f : H \rightarrow H$ denote the linear map $T_f(h) = f.h$. If H is a Hilbert module over F , then by the continuity in the second variable we have that T_f is bounded.

Definition 1.2. Let H be a Hilbert module over F . Then the *module bound of H* , is

$$K_F(H) = \inf\{K : \|T_f\| \leq K \|f\| \text{ for all } f \text{ in } A\}.$$

We call H *contractive* if $K_F(H) \leq 1$.

If H is a Hilbert module over A , then a set $\{h_\delta\}_{\delta \in \Gamma} \subset H$ is called a *generating set* for H if finite linear sums of the form

$$\sum_i f_i.h_{\delta_i}, f_i \in A, \delta_i \in \Gamma$$

are dense in H .

Definition 1.3. If H is a Hilbert module over A , then $\text{rank}_A(H)$, the *rank of H over A* , is the minimum cardinality of a generating set for H .

In the last few decades, the theory of Hilbert modules over function algebras has been developed by Ronald G. Douglas and Vern I. Paulsen [4].

1.2. A Functional Calculus. Let H be a Hilbert space. Recall that H^∞ is the Banach space of all (complex-valued) bounded analytic functions on the open unit disk \mathbf{D} with supremum norm [7]. A contraction T in $L(H)$ is said to be *completely nonunitary* if there is no invariant subspace K for T such that $T|_K$ is a unitary operator.

B. Sz.-Nagy and C. Foias introduced an important functional calculus for completely non-unitary contractions.

Proposition 1.4. *Let $T \in L(H)$ be a completely non-unitary contraction. Then there is a unique algebra representation Φ_T from H^∞ into $L(H)$ such that :*

- (i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
- (ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in \mathbf{D}$;
- (iii) Φ_T is continuous when H^∞ and $L(H)$ are given the weak*-topology.
- (iv) Φ_T is contractive, i.e. $\|\Phi_T(u)\| \leq \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B. Sz.-Nagy and C. Foias [7] defined the *class C_0* relative to the open unit disk \mathbf{D} consisting of completely non-unitary contractions T on H such that the kernel of Φ_T is not trivial. If $T \in L(H)$ is an operator of class C_0 , then

$$\ker \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak*-closed ideal of H^∞ , and hence there is an inner function generating $\ker \Phi_T$. The *minimal function* m_T of an operator of class C_0 is the generator of $\ker \Phi_T$. Also, m_T is uniquely determined up to a constant scalar factor of absolute value one [2]. The theory of class C_0 relative to the open unit disk has been developed by B.Sz.-Nagy, C. Foias ([7]) and H. Bercovici ([2]).

1.3. Hardy spaces. We refer to [9] for basic facts about Hardy space, and recall here the basic definitions.

Definition 1.5. The space $H^2(\Omega)$ is defined to be the space of analytic functions f on Ω such that the subharmonic function $|f|^2$ has a harmonic majorant on Ω . For a fixed $z_0 \in \Omega$, there is a norm on $H^2(\Omega)$ defined by

$$\|f\| = \inf\{u(z_0)^{1/2} : u \text{ is a harmonic majorant of } |f|^2\}.$$

Let m be harmonic measure for the point z_0 , let $L^2(\partial\Omega)$ be the L^2 -space of complex valued functions on the boundary of Ω defined with respect to m , and let $H^2(\partial\Omega)$ be the set of functions f in $L^2(\partial\Omega)$ such that $\int_{\partial\Omega} f(z)g(z)dz = 0$ for every g that is analytic in a neighborhood of the closure of Ω . If f is in $H^2(\Omega)$, then there is a function f^* in $H^2(\partial\Omega)$ such that $f(z)$ approaches $f^*(\lambda_0)$ as z approaches λ_0 nontangentially, for almost every λ_0 relative to m . The map $f \rightarrow f^*$ is an isometry from $H^2(\Omega)$ onto $H^2(\partial\Omega)$. In this way, $H^2(\Omega)$ can be viewed as a closed subspace of $L^2(\partial\Omega)$.

A function f defined on Ω is in $H^\infty(\Omega)$ if it is holomorphic and bounded. $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Omega)$ and it is a Banach algebra if endowed with the supremum norm. Finally, the mapping $f \rightarrow f^*$ is an isometry of $H^\infty(\Omega)$ onto a weak*-closed subalgebra of $L^\infty(\partial\Omega)$.

1.4. C_0 -operators relative to Ω . We will present in this section the definition of C_0 -operators relative to Ω . Reference to this material is found in Zucchi [10].

Let H be a Hilbert space and K_1 be a compact subset of the complex plane. If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, for $r = p/q$ a rational function with poles off K_1 , we can define an operator $r(T)$ by $q(T)^{-1}p(T)$.

Definition 1.6. If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, we say that K_1 is a *spectral set* for the operator T if $\|r(T)\| \leq \max\{|r(z)| : z \in K_1\}$, whenever r is a rational function with poles off K_1 .

If $T \in L(H)$ is an operator with $\overline{\Omega}$ as a spectral set and with no normal summand with spectrum in $\partial\Omega$, i.e., T has no reducing subspace $M \subseteq H$ such that $T|_M$ is normal and $\sigma(T|_M) \subseteq \partial\Omega$, then we say that T satisfies *hypothesis (h)*.

Proposition 1.7. ([20], Theorem 3.1.4) *Let $T \in L(H)$ be an operator satisfying hypothesis (h). Then there is a unique algebra representation Ψ_T from $H^\infty(\Omega)$ into $L(H)$ such that :*

- (i) $\Psi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
- (ii) $\Psi_T(g) = T$, where $g(z) = z$ for all $z \in \Omega$;
- (iii) Ψ_T is continuous when $H^\infty(\Omega)$ and $L(H)$ are given the weak*-topology.
- (iv) Ψ_T is contractive, i.e., $\|\Psi_T(f)\| \leq \|f\|$ for all $f \in H^\infty(\Omega)$.

From now on we will indicate $\Psi_T(f)$ by $f(T)$ for all $f \in H^\infty(\Omega)$.

Definition 1.8. An operator T satisfying hypothesis (h) is said to be of *class C_0 relative to Ω* if there exists $u \in H^\infty(\Omega) \setminus \{0\}$ such that $u(T) = 0$.

2. ALGEBRAIC ELEMENTS WITH RESPECT TO AN OPERATOR SATISFYING HYPOTHESIS (h)

Every operator T satisfying hypothesis (h) is not a C_0 -operator relative to Ω , and so we provide the following definition.

Definition 2.1. Let $T \in L(H)$ be an operator satisfying hypothesis (h). An element h of H is said to be *algebraic with respect to T* provided that $\theta(T)h = 0$ for some $\theta \in H^\infty(\Omega) \setminus \{0\}$.

If not, h is said to be *transcendental with respect to T* .

If A is a closed subspace of H generated by $\{a_i \in H : i = 1, 2, 3, \dots\}$, then A will be denoted by $\bigvee_{n=1}^\infty a_i$.

Proposition 2.2. Let $T \in L(H)$ be an operator satisfying hypothesis (h).

- (a) If $h \in H$ is algebraic with respect to T , then so is any element in $\bigvee_{n=0}^\infty T^n h$.
- (b) If $h \in H$ is transcendental with respect to T , then so is $T^n h$ for any $n = 0, 1, 2, \dots$.

Proof. (a) Let $\theta \in H^\infty(\Omega) \setminus \{0\}$ such that $\theta(T)h = 0$.

Then for any $n = 0, 1, 2, \dots$,

$$\theta(T)(T^n h) = T^n(\theta(T)h) = 0.$$

It follows that $\theta(T)h' = 0$ for any $h' \in \bigvee_{n=0}^\infty T^n h$.

(b) Suppose that $T^k h$ is algebraic with respect to T for some $k > 0$. Thus there is $f \in H^\infty(\Omega) \setminus \{0\}$ such that $f(T)T^k h = 0$.

Let $f_1(z) = z^k f(z)$ for $z \in \mathbf{D}$. Then $f_1 \in H^\infty(\Omega) \setminus \{0\}$ and

$$f_1(T)h = T^k f(T)h = f(T)T^k h = 0$$

which contradicts to the fact that h is transcendental with respect to T . \square

Note that T^0 denote the identity operator on H .

By Theorem 1 in [8], if $h \in H$ is algebraic with respect to T , then there is an inner function $m_h \in H^\infty(\Omega)$ such that $m_h(T)h = 0$ and m_h is said to be a *minimal function* of h with respect to T .

Theorem 2.3. Let $T \in L(H)$ be an operator satisfying hypothesis (h), and $B = \{h \in H : h \text{ is algebraic with respect to } T\}$.

- (a) If $M = \{h_i : i = 1, 2, \dots, k\} (k < \infty)$ is contained in B , then so is $\bigvee_{n=0}^\infty T^n M$.
- (b) B is a subspace of H .

Proof. (a) By Proposition 2.2 (a),

$$(2.1) \quad m_{h_i}(T)(T^n h_i) = 0$$

for any $i = 1, \dots, k$ and $n = 0, 1, 2, \dots$.

Let $\theta = m_{h_1} \cdots m_{h_k}$. Then $\theta \in H^\infty(\Omega) \setminus \{0\}$, and $\theta = \theta_i m_{h_i}$ for some $\theta_i \in H^\infty(\Omega) \setminus \{0\}$. Thus, by equation (2.1),

$$(2.2) \quad \theta(T)(T^n h_i) = \theta_i(T) m_{h_i}(T)(T^n h_i) = 0$$

for any $i = 1, \dots, k$ and $n = 0, 1, 2, \dots$.

If $x \in \bigvee_{n=0}^\infty T^n M$, then there is a sequence $\{x_n\}_{n=1}^\infty$ such that

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } x_n = \sum_{i=1}^k a_{n,i} P_{n,i}(T) h_i$$

for some $a_{n,i} \in \mathbb{C}$ and a polynomial $P_{n,i}$. Then, equation (2.2) implies that

$$\theta(T)(x_n) = \theta(T)(\sum_{i=1}^k a_{n,i} P_{n,i}(T) h_i) = \sum_{i=1}^k a_{n,i} \theta(T) P_{n,i}(T) h_i = 0.$$

It follows that $\theta(T)(x) = 0$ for any $x \in \bigvee_{n=0}^\infty T^n M$. Thus $x \in B$.

(b) Clearly, $0 \in B$. For h_1 and h_2 in B , if $m_1(T)h_1 = m_2(T)h_2 = 0$, where $m_i(i = 1, 2) \in H^\infty(\Omega) \setminus \{0\}$, then

$$(m_1 m_2)(T)(\alpha_1 h_1 + \alpha_2 h_2) = 0$$

for any $\alpha_i(i = 1, 2)$ in \mathbb{C} . Thus B is a subspace of H . □

Note that B does not need to be closed.

If T is a bounded operator on H and M is a (closed) invariant subspace for T , then we can define a bounded operator $T_M : H/M \rightarrow H/M$ defined by

$$T_M([h]) = [Th]$$

where H/M is the quotient space. Since M is T -invariant, T_M is well-defined. Clearly, T_M is a bounded operator on H/M .

Let $R(\Omega)$ be the algebra of rational functions with poles off $\overline{\Omega}$. We will say that a (closed) subspace N is $R(\Omega)$ -invariant (or *rationally invariant*) for an operator T if it is invariant under $u(T)$ for every $u \in R(\Omega)$.

If N is a $R(\Omega)$ -invariant subspace for an operator T satisfying hypothesis (h), then we can define $\theta(T_N) : H/N \rightarrow H/N$ by

$$\theta(T_N)([h]) = [\theta(T)h]$$

for $\theta \in H^\infty(\Omega)$ and $[h] \in H/N$. Since N is $R(\Omega)$ -invariant for the operator T , T_N is well-defined. Clearly, T_N is a bounded operator on H/N .

Definition 2.4. Let $T \in L(H)$ be an operator satisfying hypothesis (h) and M be an invariant subspace for T . An element $[h]$ of H/M is said to be *algebraic with respect to T_M* provided that $\theta(T_M)[h] = 0$ for some $\theta \in H^\infty(\Omega) \setminus \{0\}$.

If not, h is said to be *transcendental with respect to T_M* .

Proposition 2.5. Let $T \in L(H)$ be an operator satisfying hypothesis (h) and $B = \{h \in H : h \text{ is algebraic with respect to } T\}$. If B is closed, then it is $R(\Omega)$ -invariant.

Proof. Let $h \in B$ and $u \in R(\Omega)$. Then there is a nonzero function $\phi \in H^\infty(\Omega)$ such that $\phi(T)h = 0$.

It follows that $u(T)\phi(T)h = \phi(T)(u(T)h) = 0$, that is, $u(T)h \in B$. \square

Theorem 2.6. Let $T \in L(H)$ be an operator satisfying hypothesis (h) and $B = \{h \in H : h \text{ is algebraic with respect to } T\}$. If B is a closed subspace of H , then the following statements are equivalent:

- (i) $[a] \in H/B$ is algebraic with respect to T_B .
- (ii) a is algebraic with respect to T .

Proof. (i) \rightarrow (ii) Since $[a] \in H/B$ is algebraic with respect to T_B , there is a nonzero function θ_1 in $H^\infty(\Omega)$ such that $\theta_1(T)a \in B$.

It follows that

$$(2.3) \quad \theta_2(T)(\theta_1(T)a) = 0$$

for some $\theta_2 \in H^\infty(\Omega) \setminus \{0\}$.

Let $\theta_3 = \theta_1 \cdot \theta_2 \in H^\infty(\Omega) \setminus \{0\}$. Then by equation (2.3), $\theta_3(T)a = 0$, and so $a \in B$.

(ii) \rightarrow (i) If $a \in H$ is algebraic with respect to T , then there is a nonzero function θ in $H^\infty(\Omega)$ such that $\theta(T)a = 0$. Since $0 \in B$, $\theta(T_B)[a] = [\theta(T)a] = 0$. \square

Corollary 2.7. Under the same assumption as Theorem 2.6, the following statements are equivalent:

- (i) $[a] \in H/B$ is algebraic with respect to T_B .
- (ii) $[a] = [0]$.

Proof. By Theorem 2.6, it is clear. \square

Corollary 2.8. Let $T \in L(H)$ be an operator satisfying hypothesis (h) and $M \subset B$ is a $R(\Omega)$ -invariant subspace for T . Then the following statements are equivalent:

- (i) $[a] \in H/M$ is algebraic with respect to T_M .
- (ii) a is algebraic with respect to T .

Proof. It can be proven by the same way as the proof of Theorem 2.6. \square

Corollary 2.9. *Let $T \in L(H)$ be an operator satisfying hypothesis (h) and $M \subset B$ is a $R(\Omega)$ -invariant subspace for T . Then the following statements are equivalent:*

- (i) $[a] \in H/M$ is transcendental with respect to T_M .
- (ii) a is transcendental with respect to T .

We recall that if K is a Hilbert space, H is a subspace of K , $V \in L(K)$, and $T \in L(H)$, then V is said to be a *dilation* of T provided that

$$(2.4) \quad T = P_H V|_H.$$

If T and V are operators satisfying hypothesis (h) and V is a C_0 -operator relative to Ω satisfying equation (2.4), then V is said to be a C_0 -dilation of T . We will not discuss about C_0 -dilation any more in this paper.

Lemma 2.10. *Let $T \in L(H)$ be an operator satisfying hypothesis (h) and $B' = \{h \in H : h \text{ is transcendental with respect to } T\}$. If $h \in B'$, then $u(T)h \in B'$ for any $u \in R(\Omega) \setminus \{0\}$*

Proof. Suppose that there is an element h in B' such that $u(T)h$ is algebraic with respect to T for some $u \in R(\Omega) \setminus \{0\}$.

Thus there is a nonzero function $\phi \in H^\infty(\Omega)$ such that

$$(2.5) \quad \phi(T)u(T)h = 0.$$

Let $\theta = \phi \cdot u$. Then $\theta \in H^\infty(\Omega) \setminus \{0\}$ such that $\theta(T)h = 0$ by equation (2.5). It contradicts to the fact that $h \in B'$. \square

3. C_0 -HILBERT MODULES

Let H be a Hilbert space and F be a function algebra on X . Then H is a Hilbert module over F with the module action $F \times H \rightarrow H$ given by

$$f.h = f(x)h$$

for a fixed $x \in X$. Let H_x denote this Hilbert module over F . Clearly, H_x is a contractive Hilbert module over F for any $x \in X$.

Similarly, for an operator T on H satisfying hypothesis (h), if $A \subset H^\infty(\Omega)$ is a function algebra over $\overline{\Omega}$ such that every polynomial is

contained in A , then H is a Hilbert module over A with the module action $A \times H \rightarrow H$ given by

$$(3.1) \quad f.h = f(T)h.$$

In this paper, H_T denotes this Hilbert module over $A \subset H^\infty(\Omega)$. Clearly, H_T is a contractive Hilbert module over A .

In this section, A denotes a function algebra over $\overline{\Omega}$ such that every polynomial is contained in A and $A \subset H^\infty(\Omega)$.

Definition 3.1. If $T \in L(H)$ is a C_0 -operator relative to Ω , then H_T is called a C_0 -Hilbert module.

Definition 3.2. Let H and K be Hilbert modules over A . Then a module map $X : H \rightarrow K$ is a bounded, linear map satisfying $X(f.h) = f.(Xh)$ for all f in A , and h in H . Two Hilbert modules are *similar* if there is an invertible module map from H onto K , and are said to be *isomorphic* if there is a module map from H onto K which is a unitary.

Proposition 3.3. For operators $T_i (i = 1, 2)$ in $L(H)$ satisfying hypothesis (h), if T_1 and T_2 are similar operators, then H_{T_1} and H_{T_2} are similar Hilbert modules over A .

Proof. Let a module map $G : H \rightarrow H$ denote the similarity such that $GT_1 = T_2G$.

Define a linear map $Y : H_{T_1} \rightarrow H_{T_2}$ by

$$(3.2) \quad Y(f.h) = f.(Gh)$$

for $f \in A$ and $h \in H_{T_1}$.

Let $f_1.h_1 = f_2.h_2$ for $f_i \in A$ and $h_i \in H_{T_1}$. Then

$$(3.3) \quad f_1(T_1)h_1 - f_2(T_1)h_2 = 0.$$

Since $GT_1 = T_2G$, equation (3.3) implies that

$$f_1(T_2)Gh_1 = Gf_1(T_1)h_1 = Gf_2(T_1)h_2 = f_2(T_2)Gh_2.$$

It follows that $f_1.(Gh_1) = f_2.(Gh_2)$, that is, Y is well-defined.

For $h \in H_{T_1}$,

$$(3.4) \quad Y(h) = Y(1.h) = G(h).$$

By equations (3.2) and (3.4), we can conclude that Y is a module map. Since G is bijective, so is Y . □

Corollary 3.4. For operators $T_i (i = 1, 2)$ in $L(H)$ satisfying hypothesis (h), if T_1 and T_2 are unitarily equivalent, then H_{T_1} and H_{T_2} are isomorphic.

Proof. It is proven by the same way as the proof of Proposition 3.3. \square

If $T \in L(H)$ is an operator satisfying hypothesis (h) and M is a submodule of H_T over A , then by the definition of module action given in equation (3.1), we have that M is T -invariant. Furthermore, M is a invariant subspace for each operator $u(T)$ where $u \in A$.

Definition 3.5. Let $T \in L(H)$ be an operator satisfying hypothesis (h). If M is a submodule of H_T (over A) such that $T|_M : M \rightarrow M$ is a C_0 -operator relative to Ω , then M is said to be a C_0 -submodule (over A) of H_T .

Definition 3.6. Let $T \in L(H)$. If there is an element $h \in H$ which is not in the kernel of T such that $\{T^n h : n = 0, 1, 2, \dots\}$ is not linearly independent, then T is said to be *dependent*.

Theorem 3.7. If $T \in L(H)$ is a dependent operator satisfying hypothesis (h), then H_T always has a nonzero C_0 -submodule M .

Proof. Since T is dependent, there is a nonzero element h in H such that $\{T^n h : n = 0, 1, 2, \dots\} (\neq \{0\})$ is linearly dependent. It follows that

$$\sum_{n=0}^k a_n T^{i_n} h = 0,$$

for some nonzero polynomial $p(z) = \sum_{n=0}^k a_n z^{i_n} (z \in \mathbf{D})$.

Let M be the closed subspace of H generated by $\{\theta(T)h : \theta \in A\}$ and $M' = \{f \in A : f(T)h = 0\}$. Since $p \in M'$, M' is not empty.

Clearly, $f.k$ is in M for every f in A and k in M and so M is a submodule of H_T .

For any $\theta \in A$ and $f \in M'$,

$$f(T)\theta(T)h = \theta(T)f(T)h = 0.$$

It follows that $f(T)h' = 0$ for any $f \in M'$ and $h' \in M$.

Therefore, $T_0 = T|_M$ is a C_0 -operator relative to Ω , and so M is a C_0 -submodule of H_T . \square

Definition 3.8. Let $T \in L(H)$ be an operator satisfying hypothesis (h). A C_0 -submodule M of H_T over A is said to be *maximal* provided that there is no submodule M' of H_T over A such that $M \subset M'$ and $T|_{M'}$ is a C_0 -operator relative to Ω .

Corollary 3.9. Let $T \in L(H)$ be an operator satisfying hypothesis (h).

If M is a maximal C_0 -submodule of H_T and $h \in H_T \setminus M$, then $\{T^n h : n = 0, 1, 2, \dots\}$ is linearly independent.

Proof. Suppose that there is an element $h \in H_T \setminus M$ such that $\{T^n h : n = 0, 1, 2, \dots\}$ is linearly dependent.

If M' is the closed subspace of H_T generated by $\{\theta(T)h : \theta \in A\}$, then by Theorem 3.7, $T|M'$ is a C_0 -operator relative to Ω . Since $T|M$ and $T|M'$ are C_0 -operators relative to Ω , there are nonzero functions $\theta_i \in H^\infty(\Omega)$ ($i = 1, 2, \dots$) such that

$$(3.5) \quad \theta_1(T|M) = 0 \text{ and } \theta_2(T|M') = 0.$$

It follows that $\theta_1\theta_2(T|M \vee M') = 0$, that is, $T|M \vee M'$ is also a C_0 -operator relative to Ω . By maximality of M , $M \vee M' = M$ which contradicts to the fact that $h \in M' \setminus M$. \square

For an operator satisfying hypothesis (h), $T \in L(H)$, $h \in H$ is said to be *algebraic with respect to T over A* , provided that

$$\theta(T)h = 0 \text{ for some } \theta \in A \setminus \{0\}.$$

If $B = \{h \in H : h \text{ is algebraic with respect to } T \text{ over } A\}$, then we could raise the question of whether the following sentence is true or not;

If every element of H_T is algebraic with respect to T over A , then T is a C_0 -operator.

In the next Theorem, we provide a condition in which that sentence is true.

Theorem 3.10. *Let $T \in L(H)$ be an operator satisfying hypothesis (h). If H_T is a Hilbert module over A with a generating set $\{h_1, \dots, h_k\}$ ($k < \infty$) and $h_i \in B$ for $i = 1, 2, \dots, k$, then $H_T = B$ and T is a C_0 -operator.*

Proof. Since $h_i \in B$, there is a nonzero function m_i in A such that $m_i(T)h_i = 0$ for $i = 1, 2, \dots, k$. Then for any $f \in A$, $m_i(T)(f \cdot h_i) = m_i(T)f(T)h_i = f(T)m_i(T)h_i = 0$. It follows that $f \cdot h_i \in B$ for any $f \in A$.

By Theorem 2.3 (b), $\{\sum_{i=1}^k f_i \cdot h_i : f_i \in A\}$ is contained in B . Since $\{\sum_{i=1}^k f_i \cdot h_i : f_i \in A\}$ is dense in H_T , it is enough to prove that B is a closed subspace of H_T .

Let b be an element in the closure of B in the norm topology induced by the inner product defined in H_T . Then, there is a sequence $\{b_n\}_{n=1}^\infty$ in $\{\sum_{i=1}^k f_i \cdot h_i : f_i \in A\}$ such that $\lim_{n \rightarrow \infty} b_n = b$.

Define a function $m = m_1 \cdots m_k$. Then, for any $f_i \in A$,

$$(3.6) \quad m(T)\left(\sum_{i=1}^k f_i \cdot h_i\right) = m(T)\left(\sum_{i=1}^k f_i(T)h_i\right) = \sum_{i=1}^k f_i(T)m(T)h_i = 0.$$

Equation (3.6) implies that $m(T)(b_n) = 0$ for any $n = 1, 2, \dots$. Thus $m(T)b = 0$ so that $b \in B$. Therefore, $H_T = B$.

Since $m(T)b = 0$ for any $b \in B(= H_T)$, $m(T) = 0$ which proves that T is a C_0 -operator. \square

Recall that a nonzero function θ in $H(\Omega)$ is said to be *inner* if $|\theta|$ is constant almost everywhere on each component of $\partial\Omega$. Then the *Jordan block* $S(\theta)$ is an operator acting on the space $H(\theta) = H^2(\Omega) \ominus \theta H^2(\Omega)$ as follows :

$$S(\theta) = P_{H(\theta)} S|_{H(\theta)},$$

where $S \in L(H^2(\Omega))$ is defined by $(Sf)(z) = zf(z)$.

An operator $T \in L(H)$ is called a *quasiaffine transform* of an operator $T' \in L(H')$ ($T \prec T'$) if there exists an injective operator $X \in L(H, H')$ with dense range such that $T'X = XT$. T and T' are *quasisimilar* if $T \prec T'$ and $T' \prec T$.

Proposition 3.11. [10] *Let H be a separable Hilbert space and $T \in L(H)$ be an operator of class C_0 relative to Ω . Then there is a family $\{\theta_i \in H^\infty(\Omega) : i = 0, 1, 2, \dots\}$ of inner functions such that*

(i) *For $i = 1, 2, \dots$, θ_i divides θ_{i-1} , that is, $\theta_{i-1} = \theta_i \varphi$ for some $\varphi \in H^\infty(\Omega)$.*

(ii) *T is quasisimilar to $\bigoplus_{i=0}^{\infty} S(\theta_i)$.*

If $T \in L(H)$ is a C_0 -operator relative to Ω , then by Definition 1.8, $\ker \Psi_T \neq \{0\}$ and there is an inner function θ , called a *minimal function* of T , in $H^\infty(\Omega)$ such that $\ker \Psi_T = \theta H^\infty(\Omega)$ [10]. We denote by m_T the minimal function of T .

Definition 3.12. Let M be a C_0 -submodule of H_T with the following property ;

If M_1 is a C_0 -submodule of H_T such that $M \subset M_1$ and $m_{T|M_1} = m_{T|M}$, then $M = M_1$.

Then M is said to be a *locally maximal C_0 -submodule* of H_T .

Theorem 3.13. *Let H be a separable Hilbert space and $T \in L(H)$ be an operator satisfying hypothesis (h). If $B = \{h \in H : h \text{ is algebraic with respect to } T \text{ over } A\}$ is a closed subspace of H and $\text{rank}_A H_T < \infty$, then there are locally maximal C_0 -submodules $M_i (i = 0, 1, 2, \dots)$ of H_T such that*

$$M_0 \subset M_1 \subset M_2 \subset \dots$$

Proof. Let $T' = T|B$. For given element $h \in B$, we have a function $m_h \in A \setminus \{0\}$ such that

$$m_h(T)h = 0.$$

Then $m_h(T)(\varphi.h) = m_h(T)\varphi(T)h = \varphi(T)m_h(T)h = 0$ for any φ in A . Thus, B is a submodule of H_T so that $B = H_{T'}$.

Since

$$\text{rank}_A H_{T'} = \text{rank}_A B \leq \text{rank}_A H_T < \infty,$$

and every elements h in B is algebraic with respect to T' over A , Theorem 3.10 implies that $T' = T|B$ is a C_0 -operator.

Thus by Proposition 3.11, there are inner functions $\theta_i (i = 0, 1, 2, \dots)$ such that θ_{i+1} divides θ_i and $T|B$ is quasisimilar to $\bigoplus_{i=0}^{\infty} S(\theta_i)$.

For each $\theta_i (i = 0, 1, 2, \dots)$, we have a bounded linear operator $\theta_i(T) : H \rightarrow H$ such that

$$\theta_i(T)(f.h) = \theta_i(T)f(T)h = f(T)\theta_i(T)h = f.(\theta_i(T)h)$$

for any $f \in A$ and $h \in H$. Thus $\theta_i(T) (i = 0, 1, 2, \dots)$ is a module map.

It follows that $M_i = \ker(\theta_i(T))$ is a submodule of H_T and clearly, $T_i = T|M_i$ is a C_0 -operator such that $\theta_i(T_i) = 0$. Thus M_i is a C_0 -submodule of H_T .

Let $i \in \{0, 1, 2, \dots\}$ be given and M be a C_0 -submodule of H_T such that

$$(3.7) \quad M_i \subset M \text{ and } m_{T|M} = m_{T|M_i}.$$

Since $m_{T|M_i} = \theta_i$, by equation (3.7), $m_{T|M} = \theta_i$. Thus, $\theta_i(T|M) = 0$ so that

$$(3.8) \quad M \subset \ker(\theta_i(T)) = M_i.$$

From equations (3.7) and (3.8), $M = M_i$. Thus, M_i is a locally maximal C_0 -submodule of H_T for each $i = 1, 2, 3, \dots$.

Since θ_{i+1} divides θ_i for $i = 0, 1, 2, \dots$, $M_i \subset M_{i+1}$.

□

In fact, in the proof of Theorem 3.13, $T|B$ is quasisimilar to $\bigoplus_{i=0}^k S(\theta_i)$ where $k \leq \text{rank}_A H_T < \infty$. Thus, we have a finite number of locally maximal C_0 -submodules $M_i (i = 0, 1, 2, \dots, k)$.

Naturally, the following question remains : When is B closed? However, we will not discuss this question in this paper.

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